

# Generalized Bargmann functions, their growth and von Neumann lattices

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Generalized Bargmann representations which are based on generalized coherent states are considered. The growth of the corresponding analytic functions in the complex plane is studied. Results about the overcompleteness or undercompleteness of discrete sets of these generalized coherent states are given. Several examples are discussed in detail.

## I. INTRODUCTION

The Bargmann representation [1] represents quantum states by analytic functions in the complex plane. This allows the powerful theory of analytic functions to be used in a quantum mechanical context. An example of a result which can be proved only by the use of the theory of analytic functions is related to the overcompleteness or undercompleteness of discrete sets of coherent states, e.g. the von Neumann lattice of coherent states [2–15]. This is based on deep theorems relating the growth of analytic functions and the density of their zeros [16–18]. The study of the Bargmann representation determines the nature of admissible functions belonging to the Hilbert space. Furthermore the study of the paths of the zeros of the Bargmann functions under time evolution has led to important physical insight for many systems [19–27].

Many generalizations of coherent states [28–30] have been considered in the literature. In the present paper we consider the generalizations studied in [31–37] which have led to many interesting sets of generalized coherent states. We use them to define generalized Bargmann representations, which represent the various quantum states by analytic functions in the complex plane. The requirement of convergence for the scalar product in these representations determines the maximum growth of the generalized Bargmann functions and defines the corresponding Bargmann spaces. Theorems that relate the growth of analytic functions in the complex plane to the density of their zeros lead to results about the overcompleteness or undercompleteness of discrete sets of these generalized coherent states. The general theory is applied to four examples: the standard coherent states and three examples of generalized coherent states. Therefore we study explicitly three novel types of generalized Bargmann representations.

In section II we briefly review known results on the growth of analytic functions in the complex plane, and the relation to the density of their zeros. In section III we define generalized coherent states. In section IV we introduce generalized Bargmann functions and study their growth. We also show that a discrete set of these generalized coherent states is overcomplete [*resp.* undercomplete] if its density is greater [*resp.* smaller] than a critical density. In section V we discuss explicitly

four examples. We conclude in section VI with a discussion of our results.

## II. GROWTH OF ANALYTIC FUNCTIONS IN THE COMPLEX PLANE AND THE DENSITY OF THEIR ZEROS

Analytic functions are characterized by their growth and the density of their zeros and we briefly summarize these concepts [16–18]. Let  $M(R)$  be the maximum modulus of an analytic function  $f(z)$  for  $|z| = R$ . Its growth is described by the order  $r$  and the type  $s$ , which are defined as follows:

$$r = \lim_{R \rightarrow \infty} \sup \frac{\ln \ln M(R)}{\ln R}; \quad s = \lim_{R \rightarrow \infty} \sup \frac{\ln M(R)}{R^r}. \quad (1)$$

These definitions imply that  $M(R) \sim \exp(sR^r)$  as  $R$  goes to infinity (here the  $\sim$  indicates that  $M(R)$  is log-asymptotic to  $\exp(sR^r)$ ).

### Definition II.1.

- (1)  $\mathfrak{B}(r, s)$  is the set of analytic functions in the complex plane with order smaller than  $r$ , and also functions with order  $r$  and type smaller or equal to  $s$ .
- (2)  $\mathfrak{B}_1(r, s)$  is the set of analytic functions in the complex plane with order smaller than  $r$ , and also functions with order  $r$  and type smaller than  $s$ .

$\mathfrak{B}(r, s) - \mathfrak{B}_1(r, s)$  is the set of analytic functions with order  $r$  and type  $s$ .

We next consider the sequence  $\zeta_1, \dots, \zeta_N, \dots$  where  $0 < |\zeta_1| \leq |\zeta_2| \leq \dots$  and the limit is infinite (as  $N$  goes to infinity). We denote by  $n(R)$  the number of terms of this sequence within the circle  $|z| < R$ . The density of this sequence is described by the numbers

$$t = \lim_{R \rightarrow \infty} \sup \frac{\ln n(R)}{\ln R}; \quad \bar{\delta} = \lim_{R \rightarrow \infty} \sup \frac{n(R)}{R^t}; \quad \underline{\delta} = \lim_{R \rightarrow \infty} \inf \frac{n(R)}{R^t}. \quad (2)$$

In the cases considered in this paper the  $\lim_{R \rightarrow \infty} \frac{n(R)}{R^t}$  exists and therefore  $\bar{\delta} = \underline{\delta}$ . Below we will use the simpler notation  $\delta = \bar{\delta} = \underline{\delta}$ . These definitions imply that asymptotically  $n(R) \sim \delta R^t$  as  $R$  goes to infinity.

**Definition II.2.** We say that the density  $(t, \delta)$  of a sequence is smaller than  $(t_1, \delta_1)$  and we denote this by  $(t, \delta) \prec (t_1, \delta_1)$  if  $t < t_1$  and also if  $t = t_1$  and  $\delta < \delta_1$  (lexicographic order).

*Remark II.3.* The density depends only on the absolute values of  $\zeta_n$ , i.e. the sequences  $\zeta_1, \dots, \zeta_N, \dots$  and  $\zeta_1 \exp(i\theta_1), \dots, \zeta_N \exp(i\theta_N), \dots$ , where  $\theta_N$  are arbitrary real numbers, have the same density. Also if we add or subtract a finite number of complex numbers in a sequence, its density remains the same.

An example of a sequence which has density  $(t, \delta)$  is

$$\zeta_N = \exp \left[ \frac{1}{t} \ln \left( \frac{N}{\delta} \right) + i\theta_N \right] \quad (3)$$

where  $\theta_N$  are arbitrary real numbers.

The relationship between the growth of an analytic function in the complex plane and the density of its zeros is described through the following inequalities [16–18]:

$$t \leq r; \quad sr \geq \delta. \quad (4)$$

### III. GENERALIZED COHERENT STATES

Let  $\mathcal{H}$  be the Hilbert space corresponding to a Hamiltonian operator  $h$  and  $|n\rangle$  its number eigenstates. Following [31–37], we consider the generalized coherent states

$$|z; \rho\rangle = [\mathcal{N}_\rho(|z|^2)]^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\rho(n)^{1/2}} |n\rangle; \quad \mathcal{N}_\rho(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho(n)}. \quad (5)$$

Here  $\rho(n)$  is a positive function of  $n$  with  $\rho(0) = 1$ . The series in the normalization constant  $\mathcal{N}_\rho(|z|^2)$  converges within some disc  $|z| < R \leq \infty$ . **In this paper we only consider cases where  $\rho(n)$  is increasing fast enough as a function of  $n$ , so that this series converges in the whole complex plane.**

The overlap of two generalized coherent states is

$$\langle z'; \rho | z; \rho \rangle = [\mathcal{N}_\rho(|z'|^2)]^{-1/2} [\mathcal{N}_\rho(|z|^2)]^{-1/2} \mathcal{K}_\rho(z'^*, z); \quad \mathcal{K}_\rho(\zeta, z) = \sum_{n=0}^{\infty} \frac{(\zeta z)^n}{\rho(n)}. \quad (6)$$

Here  $\mathcal{K}_\rho(\zeta, z)$  is the reproducing kernel. Clearly  $\mathcal{K}_\rho(z^*, z) = \mathcal{N}_\rho(|z|^2)$ .

The choice  $\rho(n) = n!$  is an example, and leads to the standard coherent states. For  $\rho(n) = n!$  the overlap  $\langle z'; \rho | z; \rho \rangle$  is everywhere different from zero. For other  $\rho(n)$  the overlap might have zeros.

The resolution of the identity in terms of the generalized coherent states is a weak operator equality given by

$$\int_{\mathbb{C}} d^2z \widetilde{W}_\rho(|z|^2) |z; \rho\rangle \langle z; \rho| = \mathbf{1}, \quad (7)$$

where  $\widetilde{W}_\rho(x) > 0$  is a function such that

$$\int_0^\infty dx x^n W_\rho(x) = \rho(n); \quad W_\rho(x) = \frac{\pi \widetilde{W}_\rho(x)}{\mathcal{N}_\rho(x)}. \quad (8)$$

We stress that for arbitrary  $\rho(n)$  the  $\widetilde{W}_\rho(x)$  does not always exist.

The property of temporal stability states that if we act with the evolution operator corresponding to a certain Hamiltonian  $h$  on the generalized coherent states, we get other coherent states. Our generalized coherent states have this property:

$$\exp(i\tau h) |z; \rho\rangle = |z \exp(i\tau \omega); \rho\rangle. \quad (9)$$

For a study of states having the above property, see[38].

#### IV. GENERALIZED BARGMANN FUNCTIONS

Let  $|f\rangle$  be an arbitrary state in the Hilbert space  $H$ :

$$|f\rangle = \sum_{n=0}^{\infty} f_n |n\rangle; \quad \sum_{n=0}^{\infty} |f_n|^2 = 1. \quad (10)$$

We represent this state by the following analytic function in the complex plane:

$$F(z; \rho) = [\mathcal{N}_\rho(|z|^2)]^{1/2} \langle z^*; \rho | f \rangle = \sum_{n=0}^{\infty} \frac{f_n z^n}{[\rho(n)]^{1/2}}. \quad (11)$$

For  $\rho(n) = n!$  this is the standard Bargmann function. But other choices of  $\rho(n)$  lead to generalizations of the Bargmann function.

If the resolution of the identity in Eq.(7) exists, then it leads to the following expression for the scalar product of two states  $|f\rangle$  and  $|g\rangle$  represented by the functions  $F(z; \rho)$  and  $G(z; \rho)$ , correspondingly:

$$\langle g | f \rangle = (G, F) = \int_{\mathbb{C}} \frac{d^2 z}{\pi} W_\rho(|z|^2) [G(z; \rho)]^* F(z; \rho) = \sum_{n=0}^{\infty} g_n^* f_n. \quad (12)$$

It is known that a pointwise bound for  $F(z; \rho)$  is  $|F(z; \rho)|^2 \leq \mathcal{K}_\rho(z^*, z)(F, F)$  [39].

As an example, we consider the generalized coherent state  $|\zeta; \rho\rangle$  which is represented by the generalized Bargmann function

$$F_{\text{coh}}(z; \rho) = [\mathcal{N}_\rho(|\zeta|^2)]^{-1/2} \mathcal{K}_\rho(\zeta, z). \quad (13)$$

**Definition IV.1.** The Bargmann space  $\mathcal{B}(W_\rho)$  consists of analytic functions in the complex plane  $F(z; \rho)$ , such that  $(F, F)$  is finite, with the scalar product given by Eq.(12).

Let  $\mathfrak{S}$  be a set of quantum states in the space  $H$ .  $\mathfrak{S}$  is called a *total* set in  $H$ , if there exists no state  $|s\rangle$  in  $H$  which is orthogonal to all states in  $\mathfrak{S}$ .

$\mathfrak{S}$  is called *undercomplete* in  $H$ , if there exists a state  $|s\rangle$  in  $H$  which is orthogonal to all states in  $\mathfrak{S}$ .

A total set  $\mathfrak{S}$  in  $H$  is called *overcomplete* in  $H$ , if there exist at least one state  $|u\rangle$  in  $\mathfrak{S}$  such that the  $\mathfrak{S} - \{|u\rangle\}$  is also a total set.

A total set  $\mathfrak{S}$  in  $H$  is called *complete* if every proper subset is not a total set. Below we will mainly use the terms overcomplete, complete and undercomplete.

**Proposition IV.2.**

(1) If

$$W_\rho(|z|^2) \sim \exp \left[ -2\mathfrak{b}(\rho) |z|^{\mathfrak{a}(\rho)} \right] \quad (14)$$

as  $|z| \rightarrow \infty$ , then the set of functions in the generalized Bargmann space  $\mathcal{B}(W_\rho)$  satisfies the

$$\mathfrak{B}_1[\mathfrak{a}(\rho), \mathfrak{b}(\rho)] \subset \mathcal{B}(W_\rho) \subset \mathfrak{B}[\mathfrak{a}(\rho), \mathfrak{b}(\rho)]. \quad (15)$$

Functions with order of growth  $\mathfrak{a}(\rho)$  and type  $\mathfrak{b}(\rho)$ , might or might not belong to  $\mathcal{B}(W_\rho)$ .

- (2) Let  $\{z_N\}$  be a sequence of complex numbers with density  $(t, \delta)$ . The set of generalized coherent states  $|z_N; \rho\rangle$  is overcomplete [resp. undercomplete] when  $(t, \delta) \succ [\mathfrak{a}(\rho), \mathfrak{b}(\rho)\mathfrak{a}(\rho)]$  [resp.  $t < \mathfrak{a}(\rho)$ ].

*Proof.*

- (1) The integral in Eq.(12) diverges for functions with order of growth greater than  $\mathfrak{a}(\rho)$ , or for functions with order equal to  $\mathfrak{a}(\rho)$  and type greater than  $\mathfrak{b}(\rho)$ . Therefore the functions in the generalized Bargmann space  $\mathcal{B}(W_\rho)$ , belong to the set  $\mathfrak{B}[\mathfrak{a}(\rho), \mathfrak{b}(\rho)]$ .

Also if the functions have order less than  $\mathfrak{a}(\rho)$  or order equal to  $\mathfrak{a}(\rho)$  and type less than  $\mathfrak{b}(\rho)$ , the integral in Eq.(12) converges. Therefore functions in the set  $\mathfrak{B}_1[\mathfrak{a}(\rho), \mathfrak{b}(\rho)]$  belong to the generalized Bargmann space  $\mathcal{B}(W_\rho)$ .

We next show with examples, that functions in  $\mathfrak{B}[\mathfrak{a}(\rho), \mathfrak{b}(\rho)] - \mathfrak{B}_1[\mathfrak{a}(\rho), \mathfrak{b}(\rho)]$  might or might not belong to  $\mathcal{B}(W_\rho)$ . We consider the special case  $W_\rho(|z|^2) = \exp(-2\lambda|z|^2)$ , i.e.,  $\mathfrak{a}(\rho) = 2$  and  $\mathfrak{b}(\rho) = \lambda > 0$ . We also consider the functions

$$F_1(z; \rho) = \frac{\exp(\lambda z^2) - 1}{z}; \quad F_2(z; \rho) = z^n \exp(\lambda z^2). \quad (16)$$

They both have growth with order 2 and type  $\lambda$  and belong to the space  $\mathfrak{B}(2, \lambda) - \mathfrak{B}_1(2, \lambda)$ . The integral of Eq.(12) converges with the first function and diverges with the second function. Therefore the first function belongs to  $\mathcal{B}(W_\rho)$ , and the second function does not belong to it.

- (2) It follows from Eq.(11) that if  $F(\zeta; \rho) = 0$  then the corresponding state  $|f\rangle$  is orthogonal to the generalized coherent state  $|\zeta^*; \rho\rangle$ .

We consider a set of generalized coherent states  $|z_N; \rho\rangle$  with density  $(t, \delta) \succ [\mathfrak{a}(\rho), \mathfrak{b}(\rho)\mathfrak{a}(\rho)]$ . If this is not a total set then there exists a function  $F(z; \rho)$  in the set  $\mathfrak{B}[\mathfrak{a}(\rho), \mathfrak{b}(\rho)]$ , which is equal to zero for all  $z_N$ . But this is not possible because it violates the relations in Eq.(4). Therefore the set of states  $|z_N; \rho\rangle$  is a total set in the space  $H$ . The same result is also true if we subtract a finite number of terms from the sequence  $\{z_N\}$ . Therefore the set of generalized coherent states  $|z_N; \rho\rangle$  is overcomplete.

We next consider a set of generalized coherent states  $|z_N; \rho\rangle$  with density of the corresponding sequence  $t < \mathfrak{a}(\rho)$ . In this case we can construct a state which is orthogonal to all  $|z_N; \rho\rangle$  [13, 15]. We use the Hadamard theorem[16–18] and consider the analytic function

$$F(z, \rho) = P(z) \exp[Q_q(z)], \quad (17)$$

where

$$P(z) = z^m \prod_{N=1}^{\infty} E(z_N, p), \quad (18)$$

$$E(A_N, 0) = 1 - \frac{z}{z_N}, \quad (19)$$

$$E(A_N, p) = \left(1 - \frac{z}{z_N}\right) \exp \left[ \frac{z}{z_N} + \frac{z^2}{2z_N^2} + \dots + \frac{z^p}{pz_N^p} \right]. \quad (20)$$

Here  $E(A_N, p)$  are the Weierstrass factors,  $Q_q(z)$  is a polynomial of degree  $q$ , and  $p$  is an integer. The maximum of  $(p, q)$  is called the *genus* of  $F(z, \rho)$  and does not exceed its order.  $m$  is a non-negative integer and it is the multiplicity of the zero at the origin. The  $z_N$  are clearly zeros of the  $F(z, \rho)$  in Eq.(17).

It remains to show that for some  $Q_q(z)$  this function belongs to the generalized Bargmann space  $\mathcal{B}(W_\rho)$ . We take  $Q_q(z) = 0$  and then the order of the growth of  $F(z, \rho) = P(z)$  is equal to  $t$  (theorem 7 in p.16 in [17]). Therefore if  $t < \mathfrak{a}(\rho)$  this function is indeed in the generalized Bargmann space  $\mathcal{B}(W_\rho)$ , and the corresponding state is orthogonal to all generalized coherent states  $|z_N; \rho\rangle$ . Consequently the set of these coherent states is undercomplete.

□

*Remark IV.3.* In the ‘boundary case’ where the density of the sequence  $\{z_n\}$  has  $t = \mathfrak{a}(\rho)$  and  $\delta \leq \mathfrak{b}(\rho)\mathfrak{a}(\rho)$ , we can not state general results. We mention some known results for special cases.

For example, when  $t = \mathfrak{a}(\rho)$  is non-integral, we consider the function  $F(z; \rho) = P(z)$ , as we did above. This function has growth with order  $t$  and its type  $s$  satisfies  $s \leq C\delta$  where  $C$  is a constant that depends on the order  $t = \mathfrak{a}(\rho)$  (p.32 in [18]). In this case for sequences  $\{z_n\}$  with  $\delta < \mathfrak{b}C^{-1}$  the function  $F(z; \rho) = P(z)$  has growth less than  $(\mathfrak{a}(\rho), \mathfrak{b}(\rho))$  and it belongs to the Bargmann space. Therefore a set of generalized coherent states  $|z_N; \rho\rangle$  with density of the corresponding sequence  $(t, \delta)$  where  $t$  is non-integral, is undercomplete if  $\delta$  is smaller than a critical value. This result is not valid for integral  $t$  (p.32 in [18]).

Another example, is the set of standard coherent states which on a von Neumann lattice with  $A = \pi$  is known to be overcomplete by one state [29]. If we subtract a finite number of coherent states we get an undercomplete set of standard coherent states which is described by the same density. This example shows two sequences with the same density (in the ‘boundary case’), with corresponding coherent states forming an overcomplete and an undercomplete set.

## V. EXAMPLES

### A. $\rho_0(n) = n!$ : standard coherent states

For the standard coherent states  $\rho_0(n) = n!$  we have

$$\mathcal{N}_{\rho_0}(|z|^2) = \exp(|z|^2); \quad W_{\rho_0}(|z|^2) = \exp(-|z|^2). \quad (21)$$

Therefore  $\mathfrak{a}(\rho_0) = 2$  and  $\mathfrak{b}(\rho_0) = 1/2$ . In this case  $\mathfrak{B}_1(2, 1/2) \subset \mathcal{B}(W_\rho) \subset \mathfrak{B}(2, 1/2)$ . The set of coherent states  $|z_n; \rho_0\rangle$  is overcomplete or undercomplete in the cases that the density  $(t, \delta)$  of the sequence  $\{z_n\}$  is  $(t, \delta) \succ (2, 1)$  or  $t < 2$ , correspondingly.

An example is the rectangular von Neumann lattice  $z_{NM} = A(N + iM)$  where  $N, M$  are integers and  $A$  is the area of each cell. In this case  $n(R) = \pi R^2/A$  and the density is described by  $t = 2$  and  $\delta = \pi/A$ . Our results show that the set of coherent states  $\{|z_{NM}; \rho_0\rangle\}$  is overcomplete for  $A < \pi$ . For this particular example, it is also known[29] that it is undercomplete for  $A > \pi$ .

We have explained in remark II.3 that instead of the lattice  $z_{NM} = A(N + iM)$  we can also use the complex numbers  $z_{NM} = A(N + iM) \exp(i\theta_{NM})$  with arbitrary phases  $\theta_{NM}$ . The angular distribution of the zeros is totally irrelevant.

Another example is the sequence

$$\zeta_N = \exp \left[ \frac{1}{t} \ln(2N) + i\theta_N \right] \quad (22)$$

where  $\theta_N$  are arbitrary real numbers. We have seen in Eq.(3) that its density is  $(t, 1/2)$ . Therefore the set of the corresponding coherent states is overcomplete for  $t > 2$  and undercomplete for  $t < 2$ . Also the sequence

$$\zeta_N = \exp \left[ \frac{1}{2} \ln \left( \frac{N}{\delta} \right) + i\theta_N \right] \quad (23)$$

where  $\theta_N$  are arbitrary real numbers, has density  $(2, \delta)$ . Therefore the set of the corresponding coherent states is overcomplete for  $\delta > 1$ .

$$\mathbf{B.} \quad \rho_1(n) = (n!)^2$$

We consider the case that  $\rho_1(n) = (n!)^2$ . It was proved in [37] that

$$\mathcal{N}_{\rho_1}(|z|^2) = I_0(2|z|); \quad W_{\rho_1}(|z|^2) = 2K_0(2|z|) \quad (24)$$

where  $I_0$  and  $K_0$  are modified Bessel functions of first and second kind, respectively. But as  $|z| \rightarrow \infty$

$$2K_0(2|z|) \sim \left( \frac{\pi}{|z|} \right)^{1/2} \exp(-2|z|) \quad (25)$$

and therefore  $\mathfrak{a}(\rho_1) = 1$  and  $\mathfrak{b}(\rho_1) = 1$ . In this case  $\mathfrak{B}_1(1, 1) \subset \mathcal{B}(W_\rho) \subset \mathfrak{B}(1, 1)$ .

The set of coherent states  $|z_N; \rho_1\rangle$  is overcomplete [*resp.*, undercomplete] when the density  $(t, \delta)$  of the sequence  $\{z_N\}$  is  $(t, \delta) \succ (1, 1)$  [*resp.*  $t < 1$ ].

An example is the **one-dimensional** lattice  $z_N = \ell N \exp(i\theta_N)$ , where  $N$  is an integer and  $\theta_N$  are arbitrary phases. In this case  $n(R) = 2R/\ell$  and the density is described by  $t = 1$  and  $\delta = 2/\ell$ . Therefore the set of coherent states  $\{|z_N; \rho_1\rangle\}$  is overcomplete for  $\ell < 2$ .

$$\mathbf{C.} \quad \rho_2(n) = \frac{(n!)^3 \pi^{1/2}}{2\Gamma(n+3/2)}$$

We consider the case  $\rho_2(n) = \frac{(n!)^3 \pi^{1/2}}{2\Gamma(n+3/2)}$ , where  $\Gamma$  denotes the Gamma function, and for which [37]

$$\mathcal{N}_{\rho_2}(|z|^2) = [I_0(|z|)]^2 + 2|z|I_0(|z|)I_1(|z|); \quad W_{\rho_2}(|z|^2) = [K_0(|z|)]^2 \quad (26)$$

But as  $|z| \rightarrow \infty$

$$[K_0(|z|)]^2 \sim \left( \frac{\pi}{2|z|} \right) \exp(-2|z|) \quad (27)$$

Therefore  $\mathfrak{a}(\rho_2) = 1$  and  $\mathfrak{b}(\rho_2) = 1$ . In this case  $\mathfrak{B}_1(1, 1) \subset \mathcal{B}(W_\rho) \subset \mathfrak{B}(1, 1)$ .

Therefore our conclusions about the overcompleteness or undercompleteness of the coherent states  $|z_N; \rho_2\rangle$  are also valid for the coherent states here. In connection with this it is interesting to compare the growth of  $\rho(n)$  in these two cases, as  $n \rightarrow \infty$ . We use the formula [40]

$$\lim_{|z| \rightarrow \infty} \frac{\Gamma(z+a)}{\Gamma(z)} \exp(-a \ln z) = 1 \quad (28)$$

and from this we conclude that as  $n \rightarrow \infty$

$$\frac{\rho_2(n)}{\rho_1(n)} = \frac{\pi^{1/2}}{2} \frac{n!}{\Gamma(n+3/2)} \sim \frac{\pi^{1/2}}{2} \exp\left(-\frac{1}{2} \ln n\right). \quad (29)$$

Therefore the  $\rho_2(n)$  considered in this subsection grows more slowly with  $n$  than the  $\rho_1(n)$  considered in the previous subsection.

$$\mathbf{D.} \quad \rho_3(n) = \frac{\Gamma(\alpha n + \beta)}{\Gamma(\beta)}$$

We consider the case

$$\rho_3(n) = \frac{\Gamma(\alpha n + \beta)}{\Gamma(\beta)}; \quad \alpha, \beta > 0. \quad (30)$$

It was proved in [34] that

$$\mathcal{N}_{\rho_3}(|z|^2) = \Gamma(\beta) E_{\alpha, \beta}(|z|^2); \quad W_{\rho_3}(|z|^2) = \frac{|z|^{\frac{2(\beta-\alpha)}{\alpha}}}{\alpha \Gamma(\beta)} \exp\left(-|z|^{\frac{2}{\alpha}}\right) \quad (31)$$

where  $E_{\alpha, \beta}(y)$  are generalized Mittag-Leffler functions [34, 41]. Therefore  $\mathfrak{a}(\rho_3) = \alpha^{-1}$  and  $\mathfrak{b}(\rho_3) = 1$ . In this case  $\mathfrak{B}_1(\alpha^{-1}, 1) \subset \mathcal{B}(W_\rho) \subset \mathfrak{B}(\alpha^{-1}, 1)$ .



The set of coherent states  $|z_N; \rho_3\rangle$  is overcomplete or undercomplete in the cases that the density  $(t, \delta)$  of the sequence  $\{z_N\}$  is  $(t, \delta) \succ (\alpha^{-1}, \alpha^{-1})$  or  $t < \alpha^{-1}$ , correspondingly. For example, the sequence

$$\zeta_N = \exp[s \ln(N) + i\theta_N] \quad (32)$$

where  $\theta_N$  are arbitrary real numbers has density  $(s^{-1}, 1)$  and the set of corresponding coherent states is overcomplete [*resp.* undercomplete] when  $s < \alpha$  or  $s = \alpha > 1$  [*resp.*  $s > \alpha$ ]. Also, the sequence

$$\zeta_N = \exp\left[\alpha \ln\left(\frac{N}{\delta}\right) + i\theta_N\right] \quad (33)$$

where  $\theta_N$  are arbitrary real numbers has density  $(\alpha^{-1}, \delta)$ . Therefore the set of corresponding coherent states is overcomplete when  $\delta > \alpha^{-1}$ .

## VI. DISCUSSION

We have used the generalized coherent states studied in [31–37] to define generalized Bargmann analytic representations in the complex plane. For these states the scalar product in Eq.(12) converges and this determines the growth of the generalized Bargmann functions. From the growth we infer the maximum density of the zeros of these functions and this in turn determines the overcompleteness or undercompleteness of discrete sets of these generalized coherent states. In addition to the standard coherent states, we studied three examples in detail in subsections VB, VC and VD. Other examples in the same spirit can also be found.

The work provides a deeper insight into the use of the theory of analytic functions in a quantum mechanical context.

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